

## ON THE COHOMOLOGY OF GUSHEL–MUKAI SIXFOLDS

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ABSTRACT. We provide a stable rationality construction for some smooth complex Gushel–Mukai varieties of dimension 6. As a consequence, we compute the integral singular cohomology of any smooth Gushel–Mukai sixfold and in particular, show that it is torsion-free.

## 1. INTRODUCTION

A smooth (complex) Gushel–Mukai (GM for short) variety of dimension  $n$  is a smooth dimensionally transverse intersection

$$X = \mathbf{CGr}(2, V_5) \cap \mathbf{P}(W) \cap Q \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5),$$

of a cone with vertex  $\mathbf{v}$  over the Grassmannian  $\mathbf{Gr}(2, V_5)$  of two-dimensional subspaces in a five-dimensional vector space  $V_5$  with a linear subspace  $\mathbf{P}(W) \simeq \mathbf{P}^{n+4}$  and a quadratic hypersurface  $Q \subset \mathbf{P}(W)$  ([DK1, Definition 2.1]).

There are two types of smooth GM varieties. If the linear space  $\mathbf{P}(W)$  does not contain the vertex  $\mathbf{v}$ , one can rewrite  $X$  as  $\mathbf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q$ , i.e.,  $X$  is a quadratic section of a linear section of the Grassmannian. If, on the contrary,  $\mathbf{P}(W)$  contains the vertex,  $\mathbf{CGr}(2, V_5) \cap \mathbf{P}(W)$  is a cone over a linear section of the Grassmannian, and since the quadric  $Q$  does not pass through  $\mathbf{v}$  (since  $X$  is smooth), the projection from  $\mathbf{v}$  identifies  $X$  with a double cover of this linear section branched along a GM variety of dimension  $n - 1$ .

GM varieties of the first type are called *ordinary* and those of the second type are called *special*. When  $n \leq 5$ , the two types of GM varieties belong to the same deformation family; when  $n = 6$ , all GM varieties are special.

Since a smooth ordinary GM variety is a complete intersection of ample divisors in the Grassmannian  $\mathbf{Gr}(2, V_5)$ , the Lefschetz Hyperplane Theorem describes all its singular cohomology groups except for the middle one and implies that these groups are all torsion-free. By a deformation argument, the same results hold for all smooth GM varieties of dimension at most 5. This argument does not work in dimension 6 since there are no ordinary GM sixfolds.

Using analogous arguments, one can determine the cohomology groups  $H^k(X, \mathbf{Z})$  of a smooth (special) GM sixfold  $X$  except for  $k = 6$  or  $7$  (see [DK2, Proposition 3.3]). Whether the (isomorphic) torsion subgroups of  $H^6(X, \mathbf{Z})$  and  $H^7(X, \mathbf{Z})$  are zero seems to be a question not easily answered by standard tools.

The goal of this paper is to show that  $H^\bullet(X, \mathbf{Z})$  is torsion-free for any smooth GM sixfold  $X$  by using a geometrical approach. It is natural to attack this question by using the fact that  $X$  is rational ([DK1, Proposition 4.2]): a good factorization of a birational

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isomorphism  $X \dashrightarrow \mathbf{P}^6$  would lead to a description of  $H^6(X, \mathbf{Z})$ . However, the birational isomorphism described in [DK1, Proposition 4.2] (and any other construction we are aware of) is quite complicated and does not lead to a simple factorization.

Our idea is to use instead a stable rationality construction. We show that a natural  $(\mathbf{P}^3 \times \mathbf{P}^1)$ -fibration over a specific GM sixfold  $X$  is dominated (birationally) by an explicit iterated blow up of the product of  $\mathbf{P}^4$  with a smooth six-dimensional quadric. This allows us to embed the group  $H^\bullet(X, \mathbf{Z})$  into a direct sum of Tate twists of  $\mathbf{Z}$ , the cohomology of a K3 surface  $Y$ , and that of an ample divisor in the Hilbert square of  $Y$ . This implies that  $H^\bullet(X, \mathbf{Z})$  is torsion-free (see Theorem 4.4 for details). By a deformation argument, the cohomology of all smooth GM sixfolds is torsion-free.

The K3 surface  $Y$  appearing above is itself a GM variety and is related to the sixfold  $X$  by a generalization ([KP, Definition 4.7]) of the duality discussed in [DK1, Section 3.6]. In fact, we work in the opposite direction: we start from a sufficiently general GM surface  $Y$  and take  $X$  to be its generalized dual GM sixfold. Then we construct  $\mathbf{P}^3$ -fibrations over  $X$  and  $Y$  that come with maps to a six-dimensional quadric with fibers mutually orthogonal linear sections of the dual Grassmannians  $\mathrm{Gr}(2, V_5)$  and  $\mathrm{Gr}(2, V_5^\vee)$  of respective codimensions 3 and 7. A smooth linear section of  $\mathrm{Gr}(2, V_5)$  of codimension 3 is a quintic del Pezzo threefold. It is a classical observation (already known to Castelnuovo) that the projectivization of the tautological bundle on it is the blow up of  $\mathbf{P}(V_5)$  along a projection of a Veronese surface. Generalizing this to mildly singular quintic del Pezzo threefolds and performing the construction in a family gives the required stable rationality construction for  $X$ .

The paper is organized as follows. In Section 2, we describe the construction of a smooth GM sixfold  $X$  from a GM surface  $Y$  and a relation between them. In particular, we construct a quintic del Pezzo fibration structure on a  $\mathbf{P}^3$ -bundle over  $X$  whose degeneration is controlled by a  $\mathbf{P}^3$ -fibration over  $Y$ . In Section 3, we describe a rationality construction for (a  $\mathbf{P}^1$ -bundle over) a family of quintic del Pezzo threefolds. In Section 4, we describe in detail the fibration constructed in Section 2 and deduce torsion-freeness of  $H^\bullet(X, \mathbf{Z})$ . We also speculate about the structure of the Chow motive of  $X$  and its relation to that of  $Y$ , and as a byproduct give a simple computation of the Hodge numbers of  $X$ .

## 2. GM SURFACES AND THEIR GENERALIZED DUAL GM SIXFOLDS

**2.1. A general GM surface.** Let  $V_5$  be a five-dimensional vector space and set  $V'_5 := V_5^\vee$ . Let  $W' \subset \bigwedge^2 V'_5$  be a subspace of codimension 3 such that

$$M' := \mathrm{Gr}(2, V'_5) \cap \mathbf{P}(W')$$

is a smooth (quintic del Pezzo) threefold and let  $Q' \subset \mathbf{P}(W')$  be a smooth quadric such that

$$Y := M' \cap Q'$$

is a smooth GM (K3) surface ([DK1, Section 2.3]).

Consider the Hilbert squares and natural maps

$$\mathrm{Hilb}^2(Y) \hookrightarrow \mathrm{Hilb}^2(M') \hookrightarrow \mathrm{Hilb}^2(\mathbf{P}(W')) \rightarrow \mathrm{Gr}(2, W')$$

(the first two maps are induced by the embeddings  $Y \hookrightarrow M' \hookrightarrow \mathbf{P}(W')$  and the last map takes a length-2 subscheme  $\xi$  to the unique line  $\langle \xi \rangle \subset \mathbf{P}(W')$  that contains it. Consider also the

isotropic Grassmannian  $\mathrm{OGr}_{Q'}(2, W') \subset \mathrm{Gr}(2, W')$  parameterizing lines in  $\mathbf{P}(W')$  contained in  $Q'$  and define

$$(1) \quad \mathrm{Hilb}_{Q'}^2(Y) := \mathrm{Hilb}^2(Y) \times_{\mathrm{Gr}(2, W')} \mathrm{OGr}_{Q'}(2, W').$$

This threefold parameterizes length-2 subschemes  $\xi$  of  $Y$  whose linear span  $\langle \xi \rangle$  is contained in  $Q'$  and it will play an important role in the construction of Section 4.

**Lemma 2.1.** *When  $Q'$  is general,  $Y = M' \cap Q'$  is a smooth surface containing neither lines nor conics and the subscheme  $\mathrm{Hilb}_{Q'}^2(Y) \subset \mathrm{Hilb}^2(Y)$  is a smooth ample divisor.*

*Proof.* Since  $Q'$  is general, the surface  $Y$  is a general polarized K3 surface of degree 10 in  $\mathbf{P}^6$ , hence contains neither lines nor conics.

Define  $\mathrm{Hilb}_{Q'}^2(M') := \mathrm{Hilb}^2(M') \times_{\mathrm{Gr}(2, W')} \mathrm{OGr}_{Q'}(2, W')$ . A bit surprisingly, we have

$$\mathrm{Hilb}_{Q'}^2(Y) = \mathrm{Hilb}_{Q'}^2(M').$$

Indeed, if  $\xi$  is a subscheme of  $M'$  such that  $\langle \xi \rangle \subset Q'$ , then  $\xi \subset M' \cap Q' = Y$ . This gives one embedding and the other is induced by the embedding  $Y \hookrightarrow M'$ .

Let  $\mathcal{P}$  be the tautological rank-2 bundle on  $\mathrm{Gr}(2, W')$ . The vector bundle  $\mathbf{S}^2 \mathcal{P}^\vee$  is generated by the space  $\mathbf{S}^2 W'^\vee$  of global sections and the zero-locus of a section  $Q' \in \mathbf{S}^2 W'^\vee$  is  $\mathrm{OGr}_{Q'}(2, W') \subset \mathrm{Gr}(2, W')$ . By base change, the same is true on  $\mathrm{Hilb}^2(M')$ , hence for  $Q'$  general, the zero-locus  $\mathrm{Hilb}_{Q'}^2(M') \subset \mathrm{Hilb}^2(M')$  is smooth of codimension 3. Thus for general  $Q'$ , the subscheme  $\mathrm{Hilb}_{Q'}^2(Y) \subset \mathrm{Hilb}^2(Y)$  is a smooth divisor.

To prove ampleness, consider the double cover

$$\sigma: \mathrm{Bl}_{\Delta(Y)}(Y \times Y) \xrightarrow{2:1} \mathrm{Hilb}^2(Y)$$

from the blow up of  $Y \times Y$  along the diagonal. Since  $\sigma$  is finite, it is enough to show that the divisor  $\sigma^*(\mathrm{Hilb}_{Q'}^2(Y))$  is ample. The dual of the pullback  $\widetilde{\mathcal{P}}$  of the bundle  $\mathcal{P}$  to the blow up  $\mathrm{Bl}_{\Delta(Y)}(Y \times Y)$  fits into an exact sequence

$$0 \rightarrow \widetilde{\mathcal{P}}^\vee \rightarrow \mathcal{O}(h_1) \oplus \mathcal{O}(h_2) \rightarrow \epsilon_* \mathcal{O}(h) \rightarrow 0,$$

where  $h_1$  and  $h_2$  are the pullbacks of the hyperplane classes of the factors of  $Y \times Y$ ,  $\epsilon$  is the embedding of the exceptional divisor  $\mathbf{P}(T_Y) \subset \mathrm{Bl}_{\Delta(Y)}(Y \times Y)$ , and  $h$  is the pullback of the hyperplane class of  $Y$  via the projection  $\mathbf{P}(T_Y) \rightarrow Y$ . In particular,

$$c_1(\widetilde{\mathcal{P}}^\vee) = h_1 + h_2 - e,$$

where  $e$  is the class of the exceptional divisor. Since  $Y$  is an intersection of quadrics and contains no lines, the map  $\mathrm{Hilb}^2(Y) \rightarrow \mathrm{Gr}(2, W')$  is an embedding. The ample class  $c_1(\mathcal{P}^\vee)$  therefore remains ample on  $\mathrm{Hilb}^2(Y)$ , hence also the class  $h_1 + h_2 - e$  on  $\mathrm{Bl}_{\Delta(Y)}(Y \times Y)$ .

Furthermore, the symmetric square of  $\widetilde{\mathcal{P}}^\vee$  fits into exact sequence

$$0 \rightarrow \mathcal{O}(h_1 + h_2 - e) \rightarrow \mathbf{S}^2 \widetilde{\mathcal{P}}^\vee \rightarrow \mathcal{O}(2h_1) \oplus \mathcal{O}(2h_2) \rightarrow \epsilon_* \mathcal{O}(2h) \rightarrow 0.$$

The image of the global section  $Q'$  of  $\mathbf{S}^2 \widetilde{\mathcal{P}}^\vee$  in  $\mathcal{O}(2h_1) \oplus \mathcal{O}(2h_2)$  vanishes (since  $Y \subset Q'$ ), hence the class of  $\sigma^*(\mathrm{Hilb}_{Q'}^2(Y))$  in  $\mathrm{Bl}_{\Delta(Y)}(Y \times Y)$  equals  $h_1 + h_2 - e$ . As we saw, this class is ample.  $\square$

In the rest of the paper, we make the following generality assumptions on the subspace  $W' \subset \Lambda^2 V_5'$  and the quadric  $Q' \subset \mathbf{P}(W')$ :

- the intersection  $M' = \text{Gr}(2, V_5') \cap \mathbf{P}(W')$  is a smooth threefold;
  - the quadratic hypersurface  $Q' \subset \mathbf{P}(W')$  is smooth;
- (2)
- the intersection  $Y = M' \cap Q'$  is a smooth surface containing neither lines, nor conics;
  - the divisor  $\text{Hilb}_{Q'}^2(Y) \subset \text{Hilb}^2(Y)$  is smooth and ample.

**2.2. Dual GM sixfold.** As in the previous section,  $W' \subset \Lambda^2 V_5' = \Lambda^2 V_5^\vee$  is a subspace of codimension 3 and  $Q' \subset \mathbf{P}(W')$  a smooth quadratic hypersurface. Considering it as a quadric (of codimension 4) in  $\mathbf{P}(\Lambda^2 V_5^\vee)$ , we denote by

$$Q_0 := Q'^\vee \subset \mathbf{P}(\Lambda^2 V_5)$$

its projective dual. It can be described as follows.

Let  $K := W'^\perp \subset \Lambda^2 V_5$  be the orthogonal of  $W'$ , so that  $\overline{W}_0 := (\Lambda^2 V_5)/K$  is isomorphic to  $W'^\vee$ . Since  $Q'$  is smooth, the corresponding quadratic form defines an isomorphism  $W' \xrightarrow{\sim} W'^\vee = \overline{W}_0$ . Its inverse is an isomorphism  $\overline{W}_0 \xrightarrow{\sim} W' = \overline{W}_0^\vee$  and thus defines a smooth quadratic hypersurface  $\overline{Q}_0 \subset \mathbf{P}(\overline{W}_0)$ . Then  $Q_0$  is the cone  $\mathbf{C}_{\mathbf{P}(K)} \overline{Q}_0$  over  $\overline{Q}_0 \subset \mathbf{P}(\Lambda^2 V_5/K)$  with vertex  $\mathbf{P}(K)$ . In particular, it is a quadratic hypersurface of corank 3 in  $\mathbf{P}(\Lambda^2 V_5)$ .

We define a GM sixfold  $X$  as the double cover

$$X \xrightarrow{2:1} \text{Gr}(2, V_5)$$

branched along the ordinary GM fivefold  $X_0 := \text{Gr}(2, V_5) \cap Q_0$ .

**Lemma 2.2.** *Under the assumptions (2),  $X$  is a smooth GM sixfold.*

*Proof.* The argument of [DK1, Proposition 3.26] shows that  $X$  is generalized dual to  $Y$  (see also [KP, Definition 4.7]). In particular, if  $A \subset \Lambda^3 V_6$  is the Lagrangian subspace corresponding to  $X$ , the Lagrangian subspace corresponding to  $Y$  is  $A^\perp \subset \Lambda^3 V_6^\vee$ . Since both  $Y$  and  $M'$  are smooth,  $A^\perp$  contains no decomposable vectors by [DK1, Theorem 3.14]. It follows that  $A$  has no decomposable vectors either, hence  $X$  is smooth, again by [DK1, Theorem 3.14].  $\square$

The GM sixfolds  $X$  constructed in this way form a family of codimension 1 in the coarse moduli space of all GM sixfolds.

**2.3. The quadrics.** According to [DK1, Lemma 2.31], the restriction from  $\mathbf{C} \oplus \Lambda^2 V_5$  to its subspace  $\Lambda^2 V_5$  defines an isomorphism between the space of quadratic equations of  $X$  in  $\mathbf{P}(\mathbf{C} \oplus \Lambda^2 V_5)$  and the space of quadratic equations of  $X_0$  in  $\mathbf{P}(\Lambda^2 V_5)$ . Let  $Q \subset \mathbf{P}(\mathbf{C} \oplus \Lambda^2 V_5)$  be the quadric that corresponds to  $Q_0$ . The kernel space of  $Q$  equals the kernel space  $K$  of  $Q_0$  so, setting  $\overline{W} := (\mathbf{C} \oplus \Lambda^2 V_5)/K = \mathbf{C} \oplus \overline{W}_0$ , we have

$$Q = \mathbf{C}_{\mathbf{P}(K)} \overline{Q},$$

where  $\overline{Q} \subset \mathbf{P}(\overline{W})$  is a non-degenerate quadric. So we have diagrams of spaces and quadrics

$$(3) \quad \begin{array}{ccc} Q_0 & \hookrightarrow & Q \\ \downarrow & & \downarrow \\ \overline{Q}_0 & \hookrightarrow & \overline{Q} \end{array} \subset \begin{array}{ccc} \mathbf{P}(\Lambda^2 V_5) & \hookrightarrow & \mathbf{P}(\mathbf{C} \oplus \Lambda^2 V_5) \\ \downarrow & & \downarrow \\ \mathbf{P}(\overline{W}_0) & \hookrightarrow & \mathbf{P}(\overline{W}), \end{array}$$

where the horizontal arrows are smooth hyperplane sections and the vertical arrows are linear projections from  $\mathbf{P}(K)$ . The quadrics  $Q' \subset \mathbf{P}(W')$  and  $\overline{Q}_0 \subset \mathbf{P}(\overline{W}_0)$  are projectively dual. Thus  $Q'$ ,  $\overline{Q}_0$ , and  $\overline{Q}$  are smooth quadric hypersurfaces in  $\mathbf{P}^6$ ,  $\mathbf{P}^6$ , and  $\mathbf{P}^7$  respectively, while  $Q_0$  and  $Q$  are quadric hypersurfaces of corank 3 in  $\mathbf{P}^9$  and  $\mathbf{P}^{10}$  respectively.

In particular,  $\overline{Q} \subset \mathbf{P}(\overline{W})$  is a smooth quadric of dimension 6. We choose one of the two connected components,  $\mathrm{OGr}_+(4, \overline{W})$ , of the corresponding orthogonal Grassmannian and we consider the corresponding connected component  $\mathrm{OFI}_+(1, 4; \overline{W})$  of the orthogonal flag variety

$$\begin{array}{ccc} & \mathrm{OFI}_+(1, 4; \overline{W}) & \\ p_{\overline{Q}} \swarrow & & \searrow q_{\overline{Q}} \\ \overline{Q} & & \mathrm{OGr}_+(4, \overline{W}), \end{array}$$

where  $p_{\overline{Q}}$  and  $q_{\overline{Q}}$  are both  $\mathbf{P}^3$ -fibrations:  $q_{\overline{Q}}$  is the projectivization of the tautological rank-4 bundle  $\mathcal{T}_4$  on  $\mathrm{OGr}_+(4, \overline{W})$  and  $p_{\overline{Q}}$  is the projectivization of the spinor bundle  $\mathcal{S}_{\overline{Q}}$ . The scheme

$$(4) \quad B := \mathrm{OGr}_+(4, \overline{W})$$

is isomorphic to a 6-dimensional quadric (it will be the base for the family of quintic del Pezzo threefolds considered later). The above diagram takes the form

$$(5) \quad \begin{array}{ccc} & \mathbf{P}(\mathcal{S}_{\overline{Q}}) = \mathbf{P}_B(\mathcal{T}_4) & \\ p_{\overline{Q}} \swarrow & & \searrow q_{\overline{Q}} \\ \overline{Q} & & B. \end{array}$$

We transfer this diagram to the other quadrics by using the maps in (3)

$$\begin{array}{ccccc} \mathbf{P}(\mathcal{S}_{\overline{Q}_0}) = \mathbf{P}_B(\mathcal{T}_3) & & \mathbf{P}(\mathcal{S}_{Q_{\mathrm{reg}}}) \hookrightarrow \mathbf{P}_B((K \otimes \mathcal{O}_B) \oplus \mathcal{T}_4) & & \\ p_{\overline{Q}_0} \swarrow & & \swarrow p_Q & & \searrow q_Q \\ \overline{Q}_0 & & Q_{\mathrm{reg}} & & B, \end{array}$$

where  $Q_{\mathrm{reg}} = Q \setminus \mathbf{P}(K)$  is the smooth locus of  $Q$ ,  $\mathcal{S}_{\overline{Q}_0}$  and  $\mathcal{S}_{Q_{\mathrm{reg}}}$  are the pullbacks of  $\mathcal{S}_{\overline{Q}}$  via the maps  $\overline{Q}_0 \hookrightarrow \overline{Q}$  and  $Q_{\mathrm{reg}} \twoheadrightarrow \overline{Q}$ , and

$$\mathcal{T}_3 = \mathcal{T}_4 \cap (\overline{W}_0 \otimes \mathcal{O}_B)$$

is a rank-3 bundle on  $B$ . This identifies  $B$  with the orthogonal Grassmannian  $\mathrm{OGr}(3, \overline{W}_0)$  of the 5-dimensional quadric  $\overline{Q}_0$ , and  $\mathcal{T}_3$  with its tautological bundle.

The maps  $p$  in the diagrams are all  $\mathbf{P}^3$ -fibrations. The map  $q_{\overline{Q}_0}$  is a  $\mathbf{P}^2$ -fibration and the map  $q_Q$  is a  $\mathbf{P}^6$ -fibration.

The projective duality between  $\overline{Q}_0$  and  $Q'$  produces a spinor bundle  $\mathcal{S}_{Q'}$  on  $Q'$  and a diagram

$$(6) \quad \begin{array}{ccc} & \mathbf{P}(\mathcal{S}_{Q'}) = \mathbf{P}_B(\mathcal{R}) & \\ p_{Q'} \swarrow & & \searrow q_{Q'} \\ Q' & & B, \end{array}$$

where

$$(7) \quad \mathcal{R} := ((K \otimes \mathcal{O}_B) \oplus \mathcal{I}_4)^\perp \subset W' \otimes \mathcal{O}_B$$

is a rank-3 subbundle and we use the natural identification  $K^\perp = W'$ .

We describe in the next lemma the restrictions of diagrams (5) and (6) to the GM varieties  $X \subset Q_{\text{reg}}$  and  $Y \subset Q'$ . We set  $\mathcal{S}_X := \mathcal{S}_{Q_{\text{reg}}}|_X$  and  $\mathcal{S}_Y := \mathcal{S}_{Q'}|_Y$ , and we denote by  $p_X$ ,  $q_X$ ,  $p_Y$ , and  $q_Y$  the restrictions of the maps  $p_Q$ ,  $q_Q$  to  $\mathbf{P}(\mathcal{S}_X)$ , and  $p_{Q'}$ ,  $q_{Q'}$  to  $\mathbf{P}(\mathcal{S}_Y)$ .

**Lemma 2.3.** *Assume that (2) holds and consider the diagram*

$$(8) \quad \begin{array}{ccccc} & \mathbf{P}(\mathcal{S}_X) & & \mathbf{P}(\mathcal{S}_Y) & \\ p_X \swarrow & & q_X \searrow & q_Y \swarrow & p_Y \searrow \\ X & & B & & Y. \end{array}$$

The maps  $p_X$  and  $p_Y$  are  $\mathbf{P}^3$ -fibrations. Moreover, if  $R_b \subset W' \subset \Lambda^2 V_5^\vee$  is the fiber of  $\mathcal{R}$  at a point  $b \in B$  and  $R_b^\perp \subset \Lambda^2 V_5$  its orthogonal, we have

$$q_X^{-1}(b) \simeq \text{Gr}(2, V_5) \cap \mathbf{P}(R_b^\perp) \quad \text{and} \quad q_Y^{-1}(b) \simeq \text{Gr}(2, V_5^\vee) \cap \mathbf{P}(R_b).$$

The map  $q_Y$  is finite and the map  $q_X$  is flat of relative dimension 3.

*Proof.* The description of the fibers is straightforward: since  $Y = \text{Gr}(2, V_5^\vee) \cap Q'$ , the fibers of  $q_Y$  are the intersections of  $\text{Gr}(2, V_5^\vee)$  with the fibers  $\mathbf{P}(R_b)$  of  $q_{Q'}$ ; the case of  $q_X$  is analogous.

Any fiber  $q_Y^{-1}(b)$  is finite: if not, since it is an intersection of quadrics in  $\mathbf{P}(R_b) \simeq \mathbf{P}^2$ , it contains a line or a conic; but it is contained in the GM surface  $Y$ , and this contradicts (2). The proper morphism  $q_Y$  is therefore finite. Finally, the map  $q_X$  is flat by Lemma 3.1 below.  $\square$

By Lemma 2.3, the fibers of the map  $\mathbf{P}(\mathcal{S}_X) \rightarrow B$  are linear sections of  $\text{Gr}(2, V_5)$  of codimension 3. A general fiber is a smooth quintic del Pezzo threefold and the discriminant locus of  $q_X$  is  $q_Y(\mathbf{P}(\mathcal{S}_Y)) \subset B$  (see [DK1, Proposition 2.22]).

### 3. FAMILIES OF QUINTIC DEL PEZZO THREEFOLDS

In this section, we discuss a stable rationality construction for general families of quintic del Pezzo threefolds which we will later apply to the family constructed in Section 2.

**3.1. Families of quintic del Pezzo threefolds.** Let  $B$  be a Cohen–Macaulay scheme and let  $\mathcal{R} \subset \Lambda^2 V_5^\vee \otimes \mathcal{O}_B$  be a rank-3 subbundle (in Section 4, we will take  $B$  to be the 6-dimensional quadric (4) and define the subbundle  $\mathcal{R}$  by (7)) with orthogonal complement

$$\mathcal{R}^\perp := \text{Ker}(\Lambda^2 V_5 \otimes \mathcal{O}_B \rightarrow \mathcal{R}^\vee) \subset \Lambda^2 V_5 \otimes \mathcal{O}_B.$$

The next lemma shows that the map

$$\mathcal{M} := \mathbf{P}_B(\mathcal{R}^\perp) \times_{\mathbf{P}(\Lambda^2 V_5)} \text{Gr}(2, V_5) \rightarrow B$$

is a family of quintic del Pezzo threefolds. For  $b \in B$ , we denote by  $R_b \subset \Lambda^2 V_5^\vee$  and  $R_b^\perp \subset \Lambda^2 V_5$  the fibers of  $\mathcal{R}$  and  $\mathcal{R}^\perp$  at  $b$ .

**Lemma 3.1.** *Assume that the intersection  $\mathbf{P}(R_b) \cap \text{Gr}(2, V_5^\vee)$  is finite for every  $b \in B$  and empty for general  $b$ . The map  $\mathcal{M} \rightarrow B$  is then flat of relative dimension 3 with general fiber a smooth quintic del Pezzo threefold.*



*Proof.* The fiber  $\mathcal{M}_b$  of  $\mathcal{M}$  over a point  $b \in B$  is  $\mathbf{P}(R_b^\perp) \cap \mathrm{Gr}(2, V_5)$ . To show that this intersection is dimensionally transverse, we choose a line  $\mathbf{P}^1 = \mathbf{P}(R') \subset \mathbf{P}(R_b) = \mathbf{P}^2$  which does not intersect  $\mathrm{Gr}(2, V_5^\vee)$  (this is possible since  $\mathbf{P}(R_b) \cap \mathrm{Gr}(2, V_5^\vee)$  is finite). By [DK1, Proposition 2.22], the intersection  $\mathrm{Gr}(2, V_5) \cap \mathbf{P}(R'^\perp)$  is then smooth and dimensionally transverse and the intersection  $\mathrm{Gr}(2, V_5) \cap \mathbf{P}(R_b^\perp)$  is a hyperplane section, hence is also dimensionally transverse. The flatness of the map  $\mathcal{M} \rightarrow B$  easily follows.

Finally, when  $\mathbf{P}(R_b) \cap \mathrm{Gr}(2, V_5^\vee)$  is empty,  $\mathcal{M}_b$  is a smooth quintic del Pezzo threefold (again by [DK1, Proposition 2.22]).  $\square$

Consider the following composition

$$(9) \quad \varphi: \mathcal{R} \hookrightarrow \bigwedge^2 V_5^\vee \otimes \mathcal{O}_{B \times \mathbf{P}(V_5)} \rightarrow \Omega_{\mathbf{P}(V_5)}(2)$$

of morphisms on  $B \times \mathbf{P}(V_5)$  (we omit the pullback notation for the projections of  $B \times \mathbf{P}(V_5)$  to the factors). As we will see later, the geometry of  $\mathcal{M}$  can be described in terms of  $\varphi$  and its degeneracy loci. In what follows, we write  $D_k(\varphi) \subset B \times \mathbf{P}(V_5)$  for the corank  $\geq k$  locus.

To analyze  $D_k(\varphi)$ , the following observation is useful. On  $\mathbf{P}(V_5)$ , there is a canonical exact sequence

$$0 \rightarrow \Omega_{\mathbf{P}(V_5)}^2(2) \rightarrow \bigwedge^2 V_5^\vee \otimes \mathcal{O}_{\mathbf{P}(V_5)} \rightarrow \Omega_{\mathbf{P}(V_5)}(2) \rightarrow 0$$

whose restriction to a point  $v \in \mathbf{P}(V_5)$  is the exact sequence

$$0 \rightarrow \bigwedge^2(v^\perp) \rightarrow \bigwedge^2 V_5^\vee \rightarrow v^\perp \rightarrow 0,$$

where the second map is the contraction with  $v$ . It follows that

$$(10) \quad \mathrm{Ker}(\varphi_{b,v}) = R_b \cap \bigwedge^2(v^\perp),$$

where  $\varphi_{b,v}$  is the fiber of  $\varphi$  at  $(b, v) \in B \times \mathbf{P}(V_5)$ .

**Lemma 3.2.** *Assume that for every point  $b \in B$ , the intersection  $\mathbf{P}(R_b) \cap \mathrm{Gr}(2, V_5^\vee)$  is finite. Then  $D_3(\varphi) = \emptyset$ .*

*Proof.* Assume  $(b, v) \in D_3(\varphi)$ . By (10), we have inclusions  $R_b \subset \bigwedge^2(v^\perp) \subset \bigwedge^2 V_5^\vee$ , hence  $\mathbf{P}(R_b) \cap \mathrm{Gr}(2, V_5^\vee) = \mathbf{P}(R_b) \cap \mathrm{Gr}(2, v^\perp)$ . Since  $\mathrm{Gr}(2, v^\perp) \subset \mathbf{P}(\bigwedge^2(v^\perp))$  is a quadric hypersurface, this intersection is a conic in  $\mathbf{P}(R_b)$ . This contradicts the assumption of finiteness; therefore,  $D_3(\varphi) = \emptyset$ .  $\square$

**3.2. A stable rationality construction.** We keep the assumptions and notation of Section 3.1. In addition, we denote by  $\mathcal{U} \subset V_5 \otimes \mathcal{O}_{\mathcal{M}}$  the pullback of the tautological rank-2 bundle on  $\mathrm{Gr}(2, V_5)$  to  $\mathcal{M}$  and consider the natural map

$$(11) \quad f: \mathbf{P}_{\mathcal{M}}(\mathcal{U}) \rightarrow B \times \mathbf{P}(V_5).$$

The next proposition shows that under appropriate assumptions on  $\varphi$ , this map is birational.

**Proposition 3.3.** *Assume  $\mathrm{codim}(D_k(\varphi)) \geq k + 1$  for all  $k \geq 1$ . The morphism  $f$  is the blow up of  $D_1(\varphi) \subset B \times \mathbf{P}(V_5)$  and is a  $\mathbf{P}^k$ -fibration over  $D_k(\varphi) \setminus D_{k+1}(\varphi)$ .*

*Proof.* Recall that  $\mathbf{P}_{\mathrm{Gr}(2, V_5)}(\mathcal{U}) \simeq \mathrm{Fl}(1, 2; V_5) \simeq \mathbf{P}(T_{\mathbf{P}(V_5)}(-2))$  and the pullback of the hyperplane class of  $\mathrm{Gr}(2, V_5)$  is the relative hyperplane class for  $\mathbf{P}(T_{\mathbf{P}(V_5)}(-2))$  over  $\mathbf{P}(V_5)$ . Therefore,

$$\mathbf{P}_{\mathcal{M}}(\mathcal{U}) \simeq \mathbf{P}_{\mathrm{Gr}(2, V_5)}(\mathcal{U}) \times_{\mathbf{P}(\bigwedge^2 V_5)} \mathbf{P}_B(\mathcal{R}^\perp) \simeq \mathbf{P}(T_{\mathbf{P}(V_5)}(-2)) \times_{\mathbf{P}(\bigwedge^2 V_5)} \mathbf{P}_B(\mathcal{R}^\perp)$$

is the zero-locus of the section of the vector bundle  $\mathcal{R}^\vee \otimes \mathcal{O}(1)$  on  $\mathbf{P}_{B \times \mathbf{P}(V_5)}(T_{\mathbf{P}(V_5)}(-2))$  induced by  $\varphi^\vee$ . The first part then follows from [K, Lemma 2.1] applied to  $\varphi^\vee$ .

The second part follows from the fact that the preimage of  $D_k(\varphi) \setminus D_{k+1}(\varphi)$  is the projectivization of  $\text{Ker}(\varphi^\vee|_{D_k(\varphi) \setminus D_{k+1}(\varphi)})$ , which is locally free of rank  $k + 1$ .  $\square$

The blow up description of Proposition 3.3 has one drawback: the center  $D_1(\varphi)$  of the blow up is usually singular. We include it in a diagram of blow ups of other simpler loci.

**Lemma 3.4.** *There is a commutative diagram*

$$(12) \quad \begin{array}{ccc} \text{Bl}_{\pi_1^{-1}(D_2(\varphi))}(\text{Bl}_{D_1(\varphi)}(B \times \mathbf{P}(V_5))) & \xlongequal{\quad} & \text{Bl}_{\pi_2^{-1}(D_1(\varphi))}(\text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))) \\ \downarrow \pi'_2 & & \downarrow \pi'_1 \\ \text{Bl}_{D_1(\varphi)}(B \times \mathbf{P}(V_5)) & & \text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5)) \\ & \searrow \pi_1 \quad \swarrow \pi_2 & \\ & B \times \mathbf{P}(V_5) & \end{array}$$

where all the maps are blow up morphisms.

*Proof.* By [SP, Tag 085Y], both varieties in the top row are canonically isomorphic to the blow up of the product of the ideals of  $D_1(\varphi)$  and  $D_2(\varphi)$  in  $B \times \mathbf{P}(V_5)$ .  $\square$

The right side of the diagram can be further simplified: since  $D_2(\varphi) \subset D_1(\varphi)$ , the ideal of  $\pi_2^{-1}(D_1(\varphi))$  is contained in some power of the ideal  $\mathcal{I}_{E_2}$  of the exceptional divisor  $E_2$  of  $\pi_2$ . Assume

$$(13) \quad \mathcal{I}_{\pi_2^{-1}(D_1(\varphi))} \not\subset \bigcap_{m=0}^{\infty} \mathcal{I}_{E_2}^m$$

and let  $m$  be the maximal integer such that  $\mathcal{I}_{\pi_2^{-1}(D_1(\varphi))} \subset \mathcal{I}_{E_2}^m$  (when  $D_2(\varphi)$  is integral,  $m$  is the multiplicity of  $D_1(\varphi)$  along  $D_2(\varphi)$ ). Since  $\mathcal{I}_{E_2}$  is an invertible ideal, we can write

$$(14) \quad \mathcal{I}_{\pi_2^{-1}(D_1(\varphi))} = \mathcal{I}' \cdot \mathcal{I}_{E_2}^m$$

for some ideal  $\mathcal{I}' \not\subset \mathcal{I}_{E_2}$ . We let  $D'_1(\varphi)$  be the subscheme of  $\text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$  defined by the ideal  $\mathcal{I}'$ . If  $D_1(\varphi)$  is integral, the subscheme  $D'_1(\varphi)$  contains the strict transform of  $D_1(\varphi)$  in  $\text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$ .

Invertible factors of an ideal do not affect the result of the blow up, hence

$$(15) \quad \text{Bl}_{\pi_2^{-1}(D_1(\varphi))}(\text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))) \simeq \text{Bl}_{D'_1(\varphi)}(\text{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))).$$

The next lemma describes (under some conditions) the restriction of  $D'_1(\varphi)$  to the exceptional divisor of the blow up  $\pi_2$ .

**Lemma 3.5.** *Assume  $D_3(\varphi) = \emptyset$ . Let  $\mathcal{K}$  and  $\mathcal{C}$  be the kernel and the cokernel sheaves of the map  $\varphi$  restricted to  $D_2(\varphi)$ , so that both are locally free of respective ranks 2 and 3. The exceptional divisor  $E_2$  of the blow up  $\pi_2$  naturally embeds into  $\mathbf{P}(\mathcal{K}^\vee \otimes \mathcal{C})$  and we have an inclusion of schemes*

$$D'_1(\varphi) \cap E_2 \subset (\mathbf{P}(\mathcal{K}^\vee) \times_{D_2(\varphi)} \mathbf{P}(\mathcal{C})) \cap E_2,$$

which is an equality if the multiplicity of  $D_1(\varphi)$  along  $D_2(\varphi)$  equals 2.



*Proof.* Take a point  $b \in D_2(\varphi)$ . Restricting to a sufficiently small neighborhood of  $b$ , we may assume that the bundles  $\mathcal{R}$  and  $\Omega_{\mathbf{P}(V_5)}(2)$  are trivial and, choosing their trivializations appropriately, that the map  $\varphi: \mathcal{R} \rightarrow \Omega_{\mathbf{P}(V_5)}(2)$  is given by a matrix

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & 0 \\ \varphi_{31} & \varphi_{32} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the regular functions  $\varphi_{ij}$  all vanish at  $b$ . In this neighborhood, the ideal of  $D_2(\varphi)$  is generated by the 6 functions  $\{\varphi_{ij}\}_{1 \leq i \leq 3, 1 \leq j \leq 2}$ . This shows that  $E_2$  embeds into a  $\mathbf{P}^5$ -bundle over  $D_2(\varphi)$  which can be identified with  $\mathbf{P}(\mathcal{K}^\vee \otimes \mathcal{C})$ . On the other hand, the ideal of  $D_1(\varphi)$  is generated in this neighborhood by the functions

$$\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}, \varphi_{11}\varphi_{32} - \varphi_{12}\varphi_{31}, \varphi_{21}\varphi_{32} - \varphi_{22}\varphi_{31}.$$

If  $\{u_{ij}\}_{1 \leq i \leq 3, 1 \leq j \leq 2}$  are the natural vertical homogeneous coordinates on the  $\mathbf{P}^5$ -bundle, the preimages of these functions are the products of

$$u_{11}u_{22} - u_{12}u_{21}, u_{11}u_{32} - u_{12}u_{31}, u_{21}u_{32} - u_{22}u_{31}$$

with the square of an equation of the exceptional divisor. These quadratic polynomials cut out in  $\mathbf{P}(\mathcal{K}^\vee \otimes \mathcal{C})$  the fiber product  $\mathbf{P}(\mathcal{K}^\vee) \times_{D_2(\varphi)} \mathbf{P}(\mathcal{C})$  and they vanish by definition on  $D'_1(\varphi)$  (and if the multiplicity equals 2, they generate the restriction of the ideal  $\mathcal{I}'$  to  $E_2$ ). The lemma follows.  $\square$

#### 4. GM SIXFOLDS

We now consider the family of quintic del Pezzo threefolds  $q_X: \mathbf{P}(\mathcal{S}_X) \rightarrow B$  constructed in Lemma 2.3, where  $B$  is a smooth 6-dimensional quadric, the map  $\varphi$  defined in (9), and its degeneracy loci  $D_k(\varphi) \subset B \times \mathbf{P}(V_5)$ . The intersections  $\mathbf{P}(R_b) \cap \mathrm{Gr}(2, V_5^\vee)$  are finite by Lemma 2.3, hence  $D_3(\varphi) = \emptyset$  by Lemma 3.2. Let us describe  $D_2(\varphi)$ .

**Lemma 4.1.** *Assume that (2) holds. There is a  $\mathbf{P}^1$ -fibration  $D_2(\varphi) \rightarrow \mathrm{Hilb}_{Q'}^2(Y)$ . In particular,  $D_2(\varphi)$  is smooth of (expected) codimension 6 in  $B \times \mathbf{P}(V_5)$ .*

*Proof.* Assume that the rank of  $\varphi$  at  $(b, v) \in B \times \mathbf{P}(V_5)$  is 1. By (10), we have

$$\dim(R_b \cap \bigwedge^2(v^\perp)) = 2.$$

The scheme  $\xi_{b,v} := \mathbf{P}(R_b \cap \bigwedge^2(v^\perp)) \cap \mathrm{Gr}(2, v^\perp)$  is the intersection in  $\mathbf{P}(\bigwedge^2(v^\perp))$  of a line and a quadric, hence is either a line or a scheme of length 2. Since it is contained in  $Y$ , the first case is impossible and we have a well-defined map

$$(16) \quad D_2(\varphi) \rightarrow \mathrm{Hilb}^2(Y), \quad (b, v) \mapsto \xi_{b,v}.$$

The line  $\langle \xi_{b,v} \rangle$  spanned by  $\xi_{b,v}$  is  $\mathbf{P}(R_b \cap \bigwedge^2(v^\perp))$ ; it is contained in  $\mathbf{P}(R_b)$ , hence in  $Q'$  by (6). Thus the map (16) factors through  $\mathrm{Hilb}_{Q'}^2(Y)$ . Furthermore, it lifts to a map

$$\delta: D_2(\varphi) \rightarrow \mathrm{Hilb}_{Q'}^2(Y) \times_{\mathrm{OGr}(2, W')} \mathrm{OFI}(2, 3; W') \quad (b, v) \mapsto (\xi_{b,v}, R_b).$$

Since  $\mathrm{OFI}(2, 3; W')$  is a  $\mathbf{P}^1$ -bundle over  $\mathrm{OGr}(2, W')$ , the lemma will follow if we show that  $\delta$  is an isomorphism. We construct an inverse.

Let  $\xi$  be a point of  $\mathrm{Hilb}_{Q'}^2(Y)$  and let  $\mathbf{P}(R) \subset Q'$  be a plane containing the line  $\langle \xi \rangle \subset Q'$ . If  $\mathcal{U}'$  is the tautological bundle on  $\mathrm{Gr}(2, V_5^\vee)$ , the evaluation map

$$V_5 = H^0(\mathrm{Gr}(2, V_5^\vee), \mathcal{U}'^\vee) \xrightarrow{\mathrm{ev}_\xi} H^0(\xi, \mathcal{U}'^\vee|_\xi) \simeq \mathbf{C}^4$$

is surjective: if not, the line  $\langle \xi \rangle$  is contained in  $\mathrm{Gr}(2, V_5^\vee)$ , and in  $Q'$  by definition of  $\mathrm{Hilb}_{Q'}^2(Y)$ , hence it is contained in  $Y$ , contradicting (2). If  $v(\xi) \in V_5$  is a generator of the kernel of  $\mathrm{ev}_\xi$ , we have  $\xi \subset \mathrm{Gr}(2, v(\xi)^\perp) \subset \mathrm{Gr}(2, V_5^\vee)$ . On the other hand, since  $B = \mathrm{OGr}(3, W')$  and  $\mathcal{R} \subset W' \otimes \mathcal{O}_B$  is the tautological subbundle, there is a unique point  $b(R) \in B$  such that  $R = R_{b(R)}$ . The intersection  $\mathbf{P}(R_{b(R)} \cap \bigwedge^2(v(\xi)^\perp))$  contains the line  $\langle \xi \rangle$ , hence  $R_{b(R)} \cap \bigwedge^2(v(\xi)^\perp)$  is at least 2-dimensional. Thus, the association  $(\xi, R) \mapsto (b(R), v(\xi)) \in B \times \mathbf{P}(V_5)$  defines a map  $\mathrm{Hilb}_{Q'}^2(Y) \times_{\mathrm{OGr}(2, W')} \mathrm{OFl}(2, 3; W') \rightarrow D_2(\varphi)$  which is inverse to  $\delta$ .  $\square$

The next step is a description of  $D_1(\varphi)$  and  $D'_1(\varphi)$ . Recall that by definition, we have  $\mathbf{P}(\mathcal{S}_Y) \subset \mathbf{P}(\mathcal{S}_{Q'}) = \mathbf{P}_B(\mathcal{R})$ .

**Proposition 4.2.** *There is a proper birational map*

$$\rho: \mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})) \rightarrow D_1(\varphi)$$

*which is an isomorphism over the complement of  $D_2(\varphi)$  and a  $\mathbf{P}^1$ -fibration over  $D_2(\varphi)$ . In particular,  $D_1(\varphi)$  is irreducible of (expected) codimension 2 in  $B \times \mathbf{P}(V_5)$ , smooth outside of  $D_2(\varphi)$ .*

*Proof.* Since  $\mathbf{P}(\mathcal{S}_Y) = p_{Q'}^{-1}(Y)$ , we have

$$\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})) = \mathbf{P}_B(\mathcal{R}) \times_{Q'} \mathrm{Bl}_Y(Q').$$

Moreover, since  $Y$  is the transversal intersection of  $Q'$  and  $\mathrm{Gr}(2, V_5^\vee)$  in  $\mathbf{P}(\bigwedge^2 V_5^\vee)$ , we have  $\mathrm{Bl}_Y(Q') = Q' \times_{\mathbf{P}(\bigwedge^2 V_5^\vee)} \mathrm{Bl}_{\mathrm{Gr}(2, V_5^\vee)}(\mathbf{P}(\bigwedge^2 V_5^\vee))$ . Finally, one checks that

$$\mathrm{Bl}_{\mathrm{Gr}(2, V_5^\vee)}(\mathbf{P}(\bigwedge^2 V_5^\vee)) \simeq \mathbf{P}(\Omega_{\mathbf{P}(V_5)}^2(2)),$$

where the blow up map is induced by the embedding  $\Omega_{\mathbf{P}(V_5)}^2(2) \hookrightarrow \bigwedge^2 V_5^\vee \otimes \mathcal{O}_{\mathbf{P}(V_5)}$ . All this implies

$$\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})) \simeq \mathbf{P}_B(\mathcal{R}) \times_{\mathbf{P}(\bigwedge^2 V_5^\vee)} \mathbf{P}(\Omega_{\mathbf{P}(V_5)}^2(2)),$$

hence  $\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R}))$  parameterizes triples  $(b, v, \eta) \in B \times \mathbf{P}(V_5) \times \mathbf{P}(\bigwedge^2 V_5^\vee)$  such that  $\eta \in \mathbf{P}(R_b \cap \bigwedge^2(v^\perp)) = \mathbf{P}(\mathrm{Ker} \varphi_{b,v})$ . This means that the map

$$\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})) \rightarrow B \times \mathbf{P}(V_5), \quad (b, v, \eta) \mapsto (b, v)$$

factors through a surjective map onto  $D_1(\varphi)$ . It is an isomorphism over  $D_1(\varphi) \setminus D_2(\varphi)$  and the scheme-theoretic preimage of  $D_2(\varphi)$  is isomorphic to the projectivization of the rank-2 vector bundle  $\mathcal{K} := \mathrm{Ker}(\varphi|_{D_2(\varphi)})$ . This proves the proposition.  $\square$

We now analyze the right side of the diagram (12). In order to use the isomorphism (15) for the blow up  $\pi'_1$ , we need a description of the subscheme  $D'_1(\varphi) \subset \mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$  defined after (14) (the assumption (13) holds in our case since the scheme  $\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$  is integral and  $D_1(\varphi)$  is non-empty). This description is provided by the next proposition.

**Proposition 4.3.** *The subscheme  $D'_1(\varphi) \subset \mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$  is isomorphic to the blow up of  $\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R}))$  along  $\rho^{-1}(D_2(\varphi))$ . It is in particular smooth and irreducible.*

*Proof.* Consider the composition

$$\mathrm{Bl}_{\rho^{-1}(D_2(\varphi))}(\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R}))) \rightarrow \mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})) \xrightarrow{\rho} D_1(\varphi) \hookrightarrow B \times \mathbf{P}(V_5).$$

The preimage of  $D_2(\varphi) \subset B \times \mathbf{P}(V_5)$  is a Cartier divisor, hence the composition lifts to a map

$$(17) \quad \mathrm{Bl}_{\rho^{-1}(D_2(\varphi))}(\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R}))) \rightarrow \mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5)).$$

By Proposition 4.2,  $\rho$  is an isomorphism over  $D_1(\varphi) \setminus D_2(\varphi)$ , hence the image of (17) is contained in the strict transform of  $D_1(\varphi)$ , hence a fortiori in  $D'_1(\varphi)$ . So, the map (17) factors through a map

$$\mathrm{Bl}_{\rho^{-1}(D_2(\varphi))}(\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R}))) \rightarrow D'_1(\varphi)$$

which is an isomorphism over the complement of the exceptional divisor  $E_2$  of the blow up  $\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$ . We now prove that this map is an isomorphism.

By Lemma 4.1 and Proposition 4.2, its source is a smooth variety. Let us check that its target is smooth as well. By Proposition 4.2,  $D'_1(\varphi)$  is smooth of codimension 2 over the complement of  $D_2(\varphi)$ . On the other hand, its intersection with  $E_2$  is by Lemma 3.5 contained in a  $(\mathbf{P}^1 \times \mathbf{P}^2)$ -fibration over  $D_2(\varphi)$  (note that in our case,  $E_2$  equals the  $\mathbf{P}^5$ -bundle over  $D_2(\varphi)$  since  $D_2(\varphi) \subset B \times \mathbf{P}(V_5)$  is smooth of codimension 6 by Lemma 4.1). In particular, the codimension of  $D'_1(\varphi) \cap E_2$  in  $E_2$  is greater than or equal to 2. But  $E_2$  is a Cartier divisor, hence the codimension of  $D'_1(\varphi) \cap E_2$  in  $E_2$  does not exceed the codimension of  $D'_1(\varphi)$  in  $\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))$ . Altogether, this shows that the codimension of  $D'_1(\varphi) \cap E_2$  in  $E_2$  is 2 and that  $D'_1(\varphi) \cap E_2$  equals the  $(\mathbf{P}^1 \times \mathbf{P}^2)$ -fibration over  $D_2(\varphi)$ . In particular, it is smooth. Since  $E_2$  is a Cartier divisor, it follows that  $D'_1(\varphi)$  is smooth along  $E_2$ . Since it is smooth outside of  $E_2$ , it is smooth everywhere.

Finally, the preimage of  $D_2(\varphi)$  in  $\mathrm{Bl}_{\rho^{-1}(D_2(\varphi))}(\mathrm{Bl}_{\mathbf{P}(\mathcal{S}_Y)}(\mathbf{P}_B(\mathcal{R})))$  is by Proposition 4.2 an irreducible divisor. It remains to note that a proper morphism between two smooth projective varieties which is an isomorphism between the complements of two irreducible divisors induces an isomorphism on the Picard groups, hence is an isomorphism.  $\square$

We can now prove our main result. We denote by  $\mathcal{U}_X$  the pullback to  $X$  of the tautological rank-2 bundle on  $\mathrm{Gr}(2, V_5)$ .

**Theorem 4.4.** *Assume that (2) holds and let  $X$  be a GM sixfold which is generalized dual to a GM surface  $Y$ , as described in Section 2.2. There is a diagram*

$$\begin{array}{ccc} & \mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))) & \\ \swarrow & & \searrow \\ \mathbf{P}(\mathcal{S}_X) \times_X \mathbf{P}_X(\mathcal{U}_X) & & \mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5)) \\ \swarrow & & \searrow \\ X & & B \times \mathbf{P}(V_5), \end{array}$$

where the leftmost map is a fibration with fiber  $\mathbf{P}^3 \times \mathbf{P}^1$  and the top left arrow is the blow up with center a  $\mathbf{P}^2$ -fibration over  $D_2(\varphi)$ .

Furthermore, the integral cohomology of  $X$  embeds into the sum of Tate twists of  $\mathbf{Z}$ ,  $H^\bullet(Y, \mathbf{Z})$ , and  $H^\bullet(\mathrm{Hilb}_{Q'}^2(Y), \mathbf{Z})$ . In particular, it is torsion-free.

*Proof.* Consider the family of quintic del Pezzo threefolds  $\mathcal{M} = \mathbf{P}(\mathcal{S}_X) \rightarrow B$  constructed in Lemma 2.3. Since the bundle  $\mathcal{U}$  on  $\mathcal{M}$  is the pullback of the bundle  $\mathcal{U}_X$  from  $X$ , we have

$$\mathbf{P}_{\mathcal{M}}(\mathcal{U}) \simeq \mathbf{P}(\mathcal{S}_X) \times_X \mathbf{P}_X(\mathcal{U}_X).$$

Let  $\varphi$  be the map of bundles on  $B \times \mathbf{P}(V_5)$  defined in (9). By Lemma 2.3 and Lemma 3.2, we have  $D_3(\varphi) = 0$ . Hence, by Lemma 4.1 and Proposition 4.2, the assumptions of Proposition 3.3 are fulfilled and we obtain

$$\mathbf{P}(\mathcal{S}_X) \times_X \mathbf{P}_X(\mathcal{U}_X) \simeq \mathrm{Bl}_{D_1(\varphi)}(B \times \mathbf{P}(V_5)).$$

Therefore, a combination of Lemma 3.4 and isomorphism (15) shows the existence of the top left arrow and proves that it is the blow up of the preimage of  $D_2(\varphi)$ . Since  $D_3(\varphi) = \emptyset$ , the latter is, by Proposition 3.3, a  $\mathbf{P}^2$ -fibration over  $D_2(\varphi)$ .

By the projective bundle formula,  $H^\bullet(X, \mathbf{Z})$  embeds into  $H^\bullet(\mathbf{P}(\mathcal{S}_X) \times_X \mathbf{P}_X(\mathcal{U}_X), \mathbf{Z})$  and by the blow up formula and the fact that  $D_2(\varphi)$  is smooth, the latter embeds into the cohomology of the top variety.

It remains to describe  $H^\bullet(\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))), \mathbf{Z})$ . We use its description as an iterated blow up. Since all blow up centers are smooth by Lemma 4.1 and Proposition 4.3, the cohomology of the top variety is a direct sum of Tate twists of  $H^\bullet(B \times \mathbf{P}(V_5))$  and of the cohomology of the blow up centers. Explicitly, we have

$$\begin{aligned} H^\bullet(\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))), \mathbf{Z}) &= H^\bullet(B \times \mathbf{P}(V_5), \mathbf{Z}) \\ &\oplus H^\bullet(D_2(\varphi), \mathbf{Z}) \otimes (\mathbf{L} \oplus \mathbf{L}^2 \oplus \mathbf{L}^3 \oplus \mathbf{L}^4 \oplus \mathbf{L}^5) \\ &\oplus H^\bullet(D'_1(\varphi), \mathbf{Z}) \otimes \mathbf{L}, \end{aligned}$$

where  $\mathbf{L}$  is the Tate module. Furthermore, we have by Lemma 4.1

$$H^\bullet(D_2(\varphi), \mathbf{Z}) = H^\bullet(\mathrm{Hilb}_{Q'}^2(Y), \mathbf{Z}) \otimes (1 \oplus \mathbf{L}).$$

Since  $D'_1(\varphi)$  is itself an iterated blow up, we have by Proposition 4.3

$$\begin{aligned} H^\bullet(D'_1(\varphi), \mathbf{Z}) &= H^\bullet(\mathbf{P}_B(\mathcal{R}), \mathbf{Z}) \\ &\oplus H^\bullet(\mathbf{P}(\mathcal{S}_Y), \mathbf{Z}) \otimes (\mathbf{L} \oplus \mathbf{L}^2) \\ &\oplus H^\bullet(D_2(\varphi), \mathbf{Z}) \otimes (1 \oplus \mathbf{L}) \otimes (\mathbf{L} \oplus \mathbf{L}^2). \end{aligned}$$

Combining all these formulas, we deduce that  $H^\bullet(\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))), \mathbf{Z})$  equals

$$\begin{aligned} &H^\bullet(B, \mathbf{Z}) \otimes (1 \oplus 2\mathbf{L} \oplus 2\mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus \mathbf{L}^4) \\ &\oplus H^\bullet(Y, \mathbf{Z}) \otimes (\mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus 2\mathbf{L}^4 \oplus 2\mathbf{L}^5 \oplus \mathbf{L}^6) \\ &\oplus H^\bullet(\mathrm{Hilb}_{Q'}^2(Y), \mathbf{Z}) \otimes (\mathbf{L} \oplus 3\mathbf{L}^2 \oplus 5\mathbf{L}^3 \oplus 5\mathbf{L}^4 \oplus 3\mathbf{L}^5 \oplus \mathbf{L}^6), \end{aligned}$$

Since  $B \simeq Q^6$ , the first is a sum of Tate twists of  $\mathbf{Z}$ .

To prove the last statement, we note that  $H^\bullet(Y, \mathbf{Z})$  is torsion-free because  $Y$  is a K3 surface. By Lemma 2.1,  $\mathrm{Hilb}_{Q'}^2(Y)$  is a smooth ample divisor in  $\mathrm{Hilb}^2(Y)$ . Since  $H^\bullet(\mathrm{Hilb}_{Q'}^2(Y), \mathbf{Z})$  is torsion-free, the same is true for  $H^\bullet(\mathrm{Hilb}_{Q'}^2(Y), \mathbf{Z})$  by the Lefschetz Hyperplane Theorem and the Universal Coefficients Theorem. This completes the proof of the theorem.  $\square$

The argument in the proof above also works at the level of Chow motives and gives an isomorphism

$$\begin{aligned} (\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5)))) &\simeq (\mathbf{M}(B) \oplus (\mathbf{M}(Y) \otimes \mathbf{L}^2)) \otimes (1 \oplus 2\mathbf{L} \oplus 2\mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus \mathbf{L}^4) \\ &\oplus \mathbf{M}(\mathrm{Hilb}_{Q'}^2(Y)) \otimes (\mathbf{L} \oplus 3\mathbf{L}^2 \oplus 5\mathbf{L}^3 \oplus 5\mathbf{L}^4 \oplus 3\mathbf{L}^5 \oplus \mathbf{L}^6), \end{aligned}$$

where  $\mathbf{M}(-)$  stands for the integral Chow motive of a variety. On the other hand, since the map  $\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5))) \rightarrow \mathbf{P}(\mathcal{S}_X) \times_X \mathbf{P}_X(\mathcal{U})$  is the blow up with center a  $\mathbf{P}^2$ -bundle over  $D_2(\varphi)$ , we have

$$\begin{aligned} \mathbf{M}(\mathrm{Bl}_{D'_1(\varphi)}(\mathrm{Bl}_{D_2(\varphi)}(B \times \mathbf{P}(V_5)))) &\simeq \mathbf{M}(X) \otimes (1 \oplus 2\mathbf{L} \oplus 2\mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus \mathbf{L}^4) \\ &\oplus \mathbf{M}(\mathrm{Hilb}_{Q'}^2(Y)) \otimes (\mathbf{L} \oplus 3\mathbf{L}^2 \oplus 5\mathbf{L}^3 \oplus 5\mathbf{L}^4 \oplus 3\mathbf{L}^5 \oplus \mathbf{L}^6). \end{aligned}$$

Comparing the two expressions, we obtain

$$(18) \quad \mathbf{M}(X) \otimes \mathbf{M}_1 \oplus \mathbf{M}_2 \simeq (\mathbf{M}(B) \oplus (\mathbf{M}(Y) \otimes \mathbf{L}^2)) \otimes \mathbf{M}_1 \oplus \mathbf{M}_2,$$

where  $M_1 = 1 \oplus 2L \oplus 2L^2 \oplus 2L^3 \oplus L^4$  and  $M_2 = M(\text{Hilb}_Q^2(Y)) \otimes (L \oplus 3L^2 \oplus 5L^3 \oplus 5L^4 \oplus 3L^5 \oplus L^6)$ . It would be interesting to understand whether the summand  $M_2$  and the factor  $M_1$  can be canceled out, providing an isomorphism of motives  $M(X) \simeq M(B) \oplus (M(Y) \otimes L^2)$ ?

In any case, any realization functor with values in a semisimple category such that the realization of  $M_2$  is not a zero divisor, when evaluated on  $M(X)$ , satisfies the above equality. For instance, we have the following result.

**Corollary 4.5.** *The Hodge numbers of any smooth GM sixfold  $X$  satisfy*

$$(19) \quad h^{p,q}(X) = h^{p,q}(B) + h^{p-2,q-2}(Y).$$

*In particular, the Hodge diamond of a smooth GM sixfold is*

$$\begin{array}{ccccccccccc} & & & & 1 & & & & & & \\ & & & 0 & & 0 & & & & & \\ & & 0 & & 0 & 1 & & 0 & & & \\ & & 0 & 0 & & 0 & 0 & & 0 & & \\ & 0 & & 0 & 0 & 2 & & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & & 22 & & 1 & & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & 2 & & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & 0 & 1 & & 0 & & & & \\ & & & 0 & & 0 & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 1 & & & & \end{array}$$

*Proof.* For the GM sixfolds constructed in Lemma 2.2, the equality (19) follows from (18) by considering the Hodge realization functor; for arbitrary GM sixfolds, it follows by a deformation argument. It remains to note that  $B$  is a 6-dimensional quadric to obtain all the Hodge numbers.  $\square$

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